

Collapsing strange quark matter in Vaidya Geometry

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Exact solutions of the gravitational field equations for a mixture of a null charged strange quark fluid and radiation are obtained in a Vaidya space-time. The conditions for the formation of a naked singularity are analyzed by considering the behavior of radial geodesics originating from the central singularity.

I. INTRODUCTION

It is generally believed that strange quark matter, consisting of u , d and s quarks, is the most energetically favorable state of baryon matter. Witten [1] suggested that there are two ways of formation of the strange matter: the quark-hadron phase transition in the early universe and the conversion of neutron stars into strange ones at ultrahigh densities. Some theories of strong interactions, e.g. quark bag models, suppose that the breaking of physical vacuum takes place inside hadrons. As a result the vacuum energy densities inside and outside a hadron become essentially different and the vacuum pressure on a bag wall equilibrates the pressure of quarks thus stabilizing the system.

There are several proposed mechanisms for the formation of quark stars. Quark stars are expected to form during the collapse of the core of a massive star after the supernova explosion as a result of a first or second order phase transition, resulting in deconfined quark matter [2]. The proto-neutron star core or the neutron star core is a favorable environment for the conversion of ordinary matter to strange quark matter [3]. Another possibility for strange star formation is that some neutron stars in low-mass X-ray binaries can accrete sufficient mass to undergo a phase transition to become strange star [4]. Such a possibility is also supported by an unusual hard X-ray burster [5]. This mechanism has been proposed as a source of radiation emission for cosmological γ -ray bursts [6]. Other possible astrophysical phenomena related to strange stars are discussed in Ref. [2]. Hence the problem of the collapse of the strange matter is of much interest in the study of strange quark star formation, dynamics and evolution. On the other hand the theoretical understanding of the collapse is of fundamental importance in general relativity.

Usually it is considered that quark matter is formed from a Fermi gas of $3A$ quarks constituting a single color-singlet baryon with baryon number A . The theory of the equation of state of strange matter is directly based on the fundamental QCD Lagrangian [7]

$$L_{QCD} = \frac{1}{4} \sum_a F_{\mu\nu}^a F^{a\mu\nu} + \sum_{f=1}^{N_f} \bar{\psi} \left(i\gamma^\mu \partial_\mu - g\gamma^\mu A_\mu^a \frac{\lambda^a}{2} - m_f \right) \psi. \quad (1)$$

The subscript f denotes the various quark flavors u, d, s etc. and the nonlinear gluon field strength is given by [7]

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc} A_\mu^b A_\nu^c. \quad (2)$$

QCD predicts a weakening of the quark-quark interaction at short distances (or high momenta Q^2), because the one-loop series for the gluon propagator yields a running coupling constant.

Neglecting quark masses in first order perturbation theory, the equation of state for zero temperature quark matter is [1,7]

$$p = \frac{1}{3} (\rho - 4B), \quad (3)$$

where B is the difference between the energy density of the perturbative and non-perturbative QCD vacuum (the bag constant) and ρ, p are the energy density and thermodynamic pressure of the quark matter, respectively. Eq.

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(3) is essentially the equation of state of a gas of massless particles with corrections due to the QCD trace anomaly and perturbative interactions. The vacuum pressure B , which holds quark matter together, is a simple model for the long-range, confining interactions in QCD. At the surface of the star, as $p \rightarrow 0$, we have $\rho \rightarrow 4B$. The typical value of the bag constant is of the order $B = 57 \text{ MeV}/fm^3 \approx 10^{15} g/cm^3$ [2]. After the neutron matter-quark matter phase transition (which is supposed to take place in the dense core of neutron stars) the energy density of strange matter is $\rho \approx 5 \times 10^{14} g/cm^3$ [1]. Therefore quark matter always satisfy the condition $p \geq 0$.

It is the purpose of the present Letter to obtain an exact non-static solution of the Einstein field equations for a collapsing charged null strange fluid in a Vaidya space-time and to study some of its singularity properties. The Vaidya geometry permitting the incorporations of the effects of radiation offers a more realistic background than static geometries, where all back reaction is ignored. The exact solution obtained represents the generalization, for strange matter, of the collapsing solutions previously obtained by Vaidya [8], Bonnor and Vaidya [9], Lake and Zannias [10] and Husain [11]. For a general method of obtaining spherically symmetric solutions in Vaidya geometry, see Ref. [12]. The structure and properties of singularities in the gravitational collapse in Vaidya space-times have been analyzed in Refs. [13-17].

The present Letter is organized as follows. In Section II the gravitational field equations are written down and the general solution is obtained. The conditions of formation of a naked singularity are analyzed in Section III. In Section IV we discuss and conclude our results.

II. SPHERICAL COLLAPSE OF THE STRANGE QUARK NULL FLUID

In ingoing Bondi coordinates (u, r, θ, φ) and with advanced Eddington time coordinate $u = t + r$ (with $r \geq 0$ the radial coordinate and r decreasing towards the future) the line element describing the radial collapse of a coherent stream of charged strange matter is

$$ds^2 = - \left[1 - \frac{2m(u, r)}{r} \right] du^2 + 2dudr + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (4)$$

$m(u, r)$ is the mass function and gives the gravitational mass within a given radius r .

Equilibrium configuration of massless strange quark matter has equal numbers of u , d and s quarks and is electrically neutral. If the mass of the strange quark is not zero, strange quarks are depleted and the system develops a net positive charge. Since stars in their lowest energy state are supposed to be charge neutral, electrons must balance the net positive quark charge in strange matter and the condition of charge neutrality requires $\frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s - n_e = 0$, where n is the particle number density [2]. Hence quark matter is usually neutral. In principle, rather extreme astrophysical conditions could lead to a charged strange quark matter astrophysical configuration and, for the sake of generality, we shall also consider this possibility.

We shall write the matter energy-momentum tensor in the form [11,12]

$$T_{\mu\nu} = T_{\mu\nu}^{(n)} + T_{\mu\nu}^{(m)} + E_{\mu\nu}, \quad (5)$$

where

$$T_{\mu\nu}^{(n)} = \mu(u, r) l_\mu l_\nu, \quad (6)$$

is the component of the matter field that moves along the null hypersurfaces $u = \text{const.}$,

$$T_{\mu\nu}^{(m)} = (\rho + p) (l_\mu n_\nu + l_\nu n_\mu) + pg_{\mu\nu}, \quad (7)$$

represents the energy-momentum tensor of the strange quark matter and

$$E_{\mu\nu} = \frac{1}{4\pi} \left(F_{\mu\alpha} F_\nu^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right), \quad (8)$$

is the electromagnetic contribution. l_μ and n_μ are two null-vectors given by $l_\mu = \delta_{(\mu)}^{(\mu)}$ and $n_\mu = \frac{1}{2} \left[1 - \frac{2m(u, r)}{r} \right] \delta_{(\mu)}^{(\mu)} - \delta_{(\mu)}^{(1)}$, so that $l_\alpha l^\alpha = n_\alpha n^\alpha = 0$ and $l_\alpha n^\alpha = -1$ [11,12] (with $\delta_{(b)}^{(a)}$ the Kronecker symbol). The energy density and pressure in Eq. (7) have been obtained by diagonalizing the energy-momentum tensor obtained from the metric [11]. The electromagnetic tensor $F_{\mu\nu}$ obeys the Maxwell equations

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} = 0, \quad (9)$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} F^{\mu\nu}) = -4\pi j^\mu. \quad (10)$$

Without any loss of generality we take the vector potential to be [10]

$$A_\mu = \frac{q(u)}{r} \delta_{(\mu)}^{(\mu)}, \quad (11)$$

with $q(u)$ being an arbitrary function. From the Maxwell Eqs. (9), (10) it follows that the only non-vanishing components of $F_{\mu\nu}$ are $F_{ru} = -F_{ur} = \frac{q(u)}{r^2}$ and

$$E_\mu^\nu = \frac{q^2(u)}{r^4} \text{diag}(-1, 1, -1, 1). \quad (12)$$

For the energy-momentum tensor (5) the gravitational field equations take the form

$$\frac{1}{r^2} \frac{\partial m(u, r)}{\partial u} = 4\pi \mu(u, r), \quad (13)$$

$$\frac{2}{r^2} \frac{\partial m(u, r)}{\partial r} = 8\pi \rho(u, r) + \frac{q^2(u)}{r^4}, \quad (14)$$

$$-\frac{1}{r} \frac{\partial^2 m(u, r)}{\partial r^2} = 8\pi p(u, r) + \frac{q^2(u)}{r^4}. \quad (15)$$

By using the bag equation of state of strange matter given by Eq. (3), we obtain the following equation describing the dynamics of a mixture of null fluid, strange matter and electric field:

$$\frac{\partial^2 m(u, r)}{\partial r^2} = -\frac{2}{3r} \frac{\partial m(u, r)}{\partial r} - \frac{2}{3} \frac{q^2(u)}{r^3} + \frac{32\pi B}{3} r. \quad (16)$$

By means of the substitution

$$m(u, r) = m_0(u, r) + \frac{4\pi B}{3} r^3 - \frac{q^2(u)}{2r}, \quad (17)$$

we obtain a new unknown function $m_0(u, r)$, which satisfies the equation

$$\frac{\partial^2 m_0(u, r)}{\partial r^2} = -\frac{2}{3r} \frac{\partial m_0(u, r)}{\partial r}, \quad (18)$$

having the general solution

$$m_0(u, r) = C(u) r^{1/3} + F(u), \quad (19)$$

with $C(u)$ and $F(u)$ being two arbitrary functions. Hence the general solution of the gravitational field equations for collapsing strange matter in the Vaidya metric (4) is given by

$$m(u, r) = F(u) + C(u) r^{1/3} + \frac{4\pi B}{3} r^3 - \frac{q^2(u)}{2r}, \quad (20)$$

$$\mu(u, r) = \frac{1}{4\pi r^2} \left[\frac{dF(u)}{du} + \frac{dC(u)}{du} r^{1/3} - q(u) \frac{dq(u)}{du} \frac{1}{r} \right], \quad (21)$$

$$\rho(u, r) = \frac{1}{4\pi r^2} \left[\frac{1}{3} C(u) r^{-2/3} + \frac{q^2(u)}{2r^2} + 4\pi B r^2 \right], \quad (22)$$

$$p(u, r) = \frac{1}{12\pi r^2} \left[\frac{1}{3} C(u) r^{-2/3} + \frac{q^2(u)}{2r^2} - 12\pi B r^2 \right]. \quad (23)$$

The electromagnetic current follows from the Maxwell Eq. (10) and is given by

$$j^\mu = \frac{1}{4\pi r^2} \frac{dq(u)}{du} l^\mu. \quad (24)$$

For $B = 0$ and $q(u) = 0$ we obtain the solution given by Husain [11] for the null fluid pressure satisfying $p = \frac{1}{3}\rho$.

The energy-momentum tensor of the mixture of fluids under consideration belongs to the Type II fluids [11]. The energy conditions are the weak, strong and dominant energy conditions $\mu \geq 0, \rho \geq 0, p \geq 0, \rho \geq p \geq 0$ and can be satisfied by appropriately choosing the arbitrary functions $F(u)$ and $C(u)$ that characterize the injection and initial distribution of mass and $q(u)$ that describes the variation of the charge. The condition $\mu \geq 0$ is equivalent to $\frac{dm}{du} \geq 0$ and leads to

$$\frac{dF(u)}{du} + \frac{dC(u)}{du} r^{1/3} \geq q(u) \frac{dq(u)}{du} \frac{1}{r}, \quad (25)$$

imposing a simultaneous constraint on all three functions $F(u), C(u)$ and $q(u)$. For small values of r and for charged quark matter, the right hand side of Eq. (25) dominates and this energy condition could not hold. One possibility to satisfy Eq. (25) for all r is to assume that the function $q(u)$ behaves so that $\frac{dq^2(u)}{du} \rightarrow 0$ for $r \rightarrow 0$. This means that the charge in the singular point $r = 0$ is constant for all times. Alternatively, we may suppose that at extremely small radii matter is converted to strange quark matter so as to satisfy the energy condition. For neutral quark matter Eq. (25) is easily satisfied by choosing $\frac{dF(u)}{du} > 0$ and $\frac{dC(u)}{du} > 0$. To satisfy the condition $p \geq 0$ for large r , we must impose on the function $C(u)$ the constraint $C(u) \geq 36\pi B r^{7/2} \geq 0, \forall u$. With this choice the condition of the non-negative energy density is also automatically satisfied. Due to the bag equation of state (3) we always have $\rho \geq p \geq 0$ and thus the dominant energy condition holds, too.

The radii of the apparent horizon of the metric (4) are given by the solution of the equation $2m = r$. If $\lim_{u \rightarrow \infty} F(u) = F_0 = \text{const.}$, $\lim_{u \rightarrow \infty} C(u) = C_0 = \text{const.}$ and $\lim_{u \rightarrow \infty} q(u) = q_0 = \text{const.}$ then the algebraic equation determining the radii of the apparent horizons is

$$2F_0 + 2C_0 r^{1/3} + \frac{8\pi B}{3} r^3 - \frac{q_0^2}{r} = r, \quad (26)$$

which in general may have multiple solutions.

The singularities of the quark matter filled Vaidya space-time can be recognized from the behavior of the energy density and curvature scalars like e.g. $R_{\alpha\beta} R^{\alpha\beta}$ given by

$$R_{\alpha\beta} R^{\alpha\beta} = \frac{8}{r^4} \left[\frac{1}{3} C(u) r^{-8/3} + \frac{q^2(u)}{r^4} + 4\pi B \right]^2, \quad (27)$$

which diverges for $r \rightarrow 0$.

III. OUTGOING RADIAL NULL GEODESICS EQUATION

The central shell-focusing singularity (i.e. that occurring at $r = 0$) is naked if the radial null-geodesic equation admits one or more positive real root X_0 [16]. In the case of the pure Vaidya space-time it has been shown that for a linear mass function $2m(u) = \lambda u$ the singularity at $r = 0, u = 0$ is naked for $\lambda \leq \frac{1}{8}$ [17]. In order to simplify calculations we choose some particular expressions for the functions $F(u), C(u)$ and $q(u)$, e.g. $F(u) = \alpha u/2$, $C(u) = \beta u^{2/3}$ and $q(u) = \gamma u$, with $\alpha > 0, \beta > 0$ and $\gamma \geq 0$ constants. With this choice the equation of the radially outgoing, future-directed null geodesic originating at the singularity is

$$\frac{du}{dr} = \frac{2}{1 - \frac{\alpha u}{r} - \beta \left(\frac{u}{r}\right)^{2/3} - \gamma \left(\frac{u}{r}\right)^2 - \frac{8\pi B}{3} r^2}. \quad (28)$$

For the geodesic tangent to be uniquely defined and to exist at the singular point $r = 0, u = 0$ of Eq. (28) the following condition must hold [16]:

$$\lim_{u,r \rightarrow 0} \frac{u}{r} = \lim_{u,r \rightarrow 0} \frac{du}{dr} = X_0. \quad (29)$$

When the limit exists and X_0 is real and positive, there is a future directed, non-space-like geodesic originating from $r = 0, u = 0$. In this case the singularity will be, at least, locally naked. For the null geodesic Eq. (28) condition (29) leads to the following algebraic equation:

$$\gamma X_0^3 + \alpha X_0^2 + \beta X_0^{5/3} - X_0 + 2 = 0. \quad (30)$$

With the help of the substitution $X_0 = y^3$ Eq. (30) becomes a ninth order polynomial equation of the form $f(y) = \gamma y^9 + \alpha y^6 + \beta y^5 - y^3 + 2 = 0$. According to a theorem given by Poincare [18] the number of positive roots of a polynomial equation equals the number of changes in sign in the sequence of the non-negative coefficients of the polynomial $g(y) = (1+y)^k f(y)$. The Poincare criterion indicates that Eq. (30) has two positive roots. In order to give an estimate of the positive roots we shall consider the expression $g(y) = (1+y)^k f(y) = \sum_{\nu=0}^{k+9} b_k(\nu) y^\nu$, with $b_k(0) = \gamma > 0$ [18]. Let $\nu_k(1)$ denote the smallest integer for which $b_k(\nu_k(1)) \geq 0$ and $b_k(\nu_k(1)+1) < 0$. Then we obtain the numbers $\nu_k(1)$ and $\nu_k(2)$. These numbers satisfy the relations [18]

$$\frac{\nu_k(s)}{k - \nu_k(s) + 1} \leq \xi(k, \nu, s) \leq \frac{\nu_k(s) + 1}{k - \nu_k(s)}, s = 1, 2, \quad (31)$$

$$\lim_{k \rightarrow \infty} \frac{\nu_k(s)}{k - \nu_k(s) + 1} = \lim_{k \rightarrow \infty} \xi(k, \nu, s) = y_s > 0, s = 1, 2, \quad (32)$$

$$f(y_s) = 0, s = 1, 2. \quad (33)$$

Therefore it is always possible to construct a convergent sequence to obtain the positive roots of Eq.(30). In the case of the neutral quark fluid, $q(u) \equiv 0$ and Eq. (30) is reduced to a sixth order algebraic equation also having two positive roots.

IV. CONCLUSIONS

In the present paper we considered the collapse of a strange quark fluid in Vaidya geometry. The possible occurrence of a central naked singularity has also been investigated and it has been shown that, at least for a particular choice of the parameters, a naked singularity is formed. Depending on the initial distribution of density and velocity and on the constitutive nature of the collapsing matter either a black hole or a naked singularity is formed. The values of the parameters in the solution (20)-(23) determine which of these possibilities occurs. The solution describing the collapse of the quark matter is not asymptotically flat and this condition does not play any role in the formation of the naked singularity. Due to the presence of the bag constant B (playing the role of a cosmological constant, from a formal mathematical point of view) the mass function (20) gives a cosmological type metric.

Quark matter, deconfined phase of hadronic matter at high temperatures or densities may reside as a permanent component of neutron stars core or to form stable compact stellar objects [2,4,5]. In fact from a physical point of view it seems to be one of the best and more realistic candidates for the study of properties of collapsing objects. It also serves to illustrate the much richer interplay that can occur among particle physics and general relativity when more involved quantum field theoretical models are considered.

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